

Local Curvature Bound in Ricci Flow

Peng Lu

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1 Introduction. In [Pe02I] §10.3 G. Perelman gives the following theorem.

Theorem 1 *There exist $\epsilon, \delta > 0$ with the following property. Suppose $g_{ij}(t)$ is a smooth solution to the Ricci flow on $[0, (\epsilon r_0)^2]$, and assume that at $t = 0$ we have $|\text{Rm}|(x) \leq r_0^{-2}$ in $B(x_0, r_0)$, and $\text{Vol } B(x_0, r_0) \geq (1 - \delta)\omega_n r_0^n$, where ω_n is the volume of the unit ball in \mathbb{R}^n . Then the estimate $|\text{Rm}|(x, t) \leq (\epsilon r_0)^{-2}$ holds whenever $0 \leq t \leq (\epsilon r_0)^2$, $\text{dist}_t(x, x_0) < \epsilon r_0$.*

He continues: “The proof is a slight modification of the proof of theorem 10.1, and is left to the reader. A natural question is whether the assumption on the volume of the ball is superfluous.”

In this note by using the idea in the proof of Perelman’s pseudo locality theorem (Theorem 10.1 in [Pe02I]) we will show

Theorem 2 *Given n and $v_0 > 0$, there exists $\epsilon_0 > 0$ depending only on n and v_0 which has the following property. For any $r_0 > 0$ and $\epsilon \in (0, \epsilon_0]$ suppose $(M^n, g(t))$ is a complete smooth solution to the Ricci flow on $[0, (\epsilon r_0)^2]$ with bounded sectional curvature, and assume that at $t = 0$ for some $x_0 \in M$ we have curvature bound $|\text{Rm}|(x, 0) \leq r_0^{-2}$ for all $x \in B_{g(0)}(x_0, r_0)$, and volume lower bound $\text{Vol}_{g(0)}(B_{g(0)}(x_0, r_0)) \geq v_0 r_0^n$. Then $|\text{Rm}|(x, t) \leq (\epsilon r_0)^{-2}$ for all $t \in [0, (\epsilon r_0)^2]$ and $x \in B_{g(t)}(x_0, \epsilon r_0)$.*

In §2 we will give a proof of Theorem 2 using two technical lemmas which will be proved in §3. In §4 we will give an example which shows that the curvature bound in Theorem 2 is false without the assumption $\text{Vol}_{g(0)}(B_{g(0)}(x_0, r_0)) \geq v_0 r_0^n$.

2 Proof of Theorem 2. First we give a proof of Theorem 2 assuming Proposition 1 below. Then we will prove the proposition.

Proposition 1 *Given n and $v_0 > 0$, there exists $\epsilon_0 > 0$ depending only on n and v_0 which has the following property. For any $r_0 > 0$ and $\epsilon \in (0, \epsilon_0]$ suppose $(M^n, g(t))$ is a complete smooth solution to the Ricci flow on $[0, (\epsilon r_0)^2]$ with bounded*

sectional curvature, and assume that at $t = 0$ for some $x_0 \in M$ we have curvature bound $|\text{Rm}|(x, 0) \leq r_0^{-2}$ for all $x \in B_{g(0)}(x_0, r_0)$, and volume lower bound $\text{Vol}_{g(0)}(B_{g(0)}(x_0, r_0)) \geq v_0 r_0^n$. Then $|\text{Rm}|(x, t) \leq (\epsilon r_0)^{-2}$ for all $t \in [0, (\epsilon r_0)^2]$ and $x \in B_{g(0)}(x_0, e^{n-1} \epsilon r_0)$.

Proof of Theorem 2. It suffices to prove the following statement. For the solution $g(t)$ in Proposition 1 we have

$$(1) \quad B_{g(t)}(x_0, \epsilon r_0) \subset B_{g(0)}(x_0, e^{n-1} \epsilon r_0) \quad \text{for any } t \in [0, (\epsilon r_0)^2].$$

If (1) is not true, there is a point $x \in B_{g(t)}(x_0, \epsilon r_0) \setminus B_{g(0)}(x_0, e^{n-1} \epsilon r_0)$. Let $\gamma(s)$, $0 \leq s \leq s_0$, be a unit-speed minimal geodesic with respect to metric $g(t)$ with $\gamma(0) = x_0$ and $\gamma(s_0) = x$. Then $s_0 < \epsilon r_0$, and there is a $s_1 \in (0, s_0]$ such that $\gamma(s_1) \in \partial(B_{g(0)}(x_0, e^{n-1} \epsilon r_0))$ and $\gamma([0, s_1]) \subset B_{g(0)}(x_0, e^{n-1} \epsilon r_0)$. In particular, the length satisfies

$$(2) \quad L_{g(0)}(\gamma|_{[0, s_1]}) \geq e^{n-1} \epsilon r_0.$$

From the curvature bound $|\text{Rm}|(x, t) \leq (\epsilon r_0)^{-2}$ in Proposition 1 and the Ricci flow equation, we have

$$|\gamma'(s)|_{g(0)} \leq e^{(n-1)t} |\gamma'(s)|_{g(t)} \quad \text{for all } t \in [0, (\epsilon r_0)^2] \text{ and } s \in [0, s_1].$$

Hence we have

$$L_{g(0)}(\gamma|_{[0, s_1]}) \leq \int_0^{s_1} e^{(n-1)t} |\gamma'(s)|_{g(t)} ds \leq e^{(n-1)t} \cdot s_0 < e^{n-1} \epsilon r_0.$$

This contradicts with (2), and (1) is proved. The theorem is proved assuming Proposition 1. \square

In the rest of this section we give a proof of Proposition 1.

Proof of Proposition 1. After parabolically scaling the Ricci flow $g(t)$ by r_0^{-2} , it suffices to prove the proposition for $r_0 = 1$ which we assume from now on. Suppose the proposition is not true, then there are $n, v_0 > 0$, a sequence of $\epsilon_i \rightarrow 0^+$, and a sequence of complete smooth solutions to the Ricci flow $(M_i^n, g_i(t))$, $t \in [0, \epsilon_i^2]$, with bounded sectional curvature such that the following is true for each i :

- (i) $|\text{Rm}_{g_i}|(x, 0) \leq 1$ for all $x \in B_{g_i(0)}(x_{0i}, 1)$. Here $x_{0i} \in M_i$.
- (ii) $\text{Vol}_{g_i(0)}(B_{g_i(0)}(x_{0i}, 1)) \geq v_0$.
- (iii) There are $t_i \in (0, \epsilon_i^2]$ and $x_i \in B_{g_i(0)}(x_{0i}, e^{n-1} \epsilon_i)$ such that $|\text{Rm}_{g_i}|(x_i, t_i) > \epsilon_i^{-2}$.
- (iv) $\epsilon_i \leq \frac{1}{4e^{n-1}}$.

To get a contradiction from the existence of sequence $\{(M_i, g_i(t))\}$, we need the following point-picking statement whose proof is simpler than the proofs of the point-picking claims used by Perelman in [Pe02I], §10.1. Let $A_i \doteq \frac{1}{100n\epsilon_i}$.

Claim A. Fix any i , there are points $(\bar{x}_i, \bar{t}_i) \in B_{g_i(0)}(x_{0i}, (2A_i + e^{n-1})\epsilon_i) \times (0, \epsilon_i^2]$ with $\bar{Q}_i \doteq |\text{Rm}_{g_i}|(\bar{x}_i, \bar{t}_i) > \epsilon_i^{-2}$ such that

$$|\text{Rm}_{g_i}|(x, t) \leq 4\bar{Q}_i \quad \text{for all } (x, t) \in B_{g_i(0)}\left(\bar{x}_i, A_i \bar{Q}_i^{-1/2}\right) \times (0, \bar{t}_i].$$

Proof of Claim A. Let $Q_i^0 \doteq |\text{Rm}_{g_i}|(x_i, t_i)$. If (x_i, t_i) from (iii) satisfies the curvature bound of the claim, i.e.,

$$|\text{Rm}_{g_i}|(x, t) \leq 4Q_i^0 \quad \text{for all } (x, t) \in B_{g_i(0)}\left(x_i, A_i(Q_i^0)^{-1/2}\right) \times (0, t_i],$$

we choose $(\bar{x}_i, \bar{t}_i) = (x_i, t_i)$ and the claim is proved.

If (x_i, t_i) does not satisfy the curvature bound of the claim, then there is a point

$$(x_i^1, t_i^1) \in B_{g_i(0)}\left(x_i, A_i(Q_i^0)^{-1/2}\right) \times (0, t_i]$$

such that $|\text{Rm}_{g_i}|(x_i^1, t_i^1) > 4Q_i^0$. We compute using $Q_i^0 > \epsilon_i^{-2}$

$$\begin{aligned} d_{g_i(0)}(x_i^1, x_{0i}) &\leq d_{g_i(0)}(x_i, x_{0i}) + A_i(Q_i^0)^{-1/2} \\ &\leq e^{n-1}\epsilon_i + A_i\epsilon_i < (2A_i + e^{n-1})\epsilon_i. \end{aligned}$$

If (x_i^1, t_i^1) satisfies the curvature bound of the claim, we choose $(\bar{x}_i, \bar{t}_i) = (x_i^1, t_i^1)$ and the claim is proved.

If (x_i^1, t_i^1) does not satisfy the claim, let $Q_i^1 \doteq |\text{Rm}_{g_i}|(x_i^1, t_i^1)$, then there is a point

$$(x_i^2, t_i^2) \in B_{g_i(0)}\left(x_i^1, A_i(Q_i^1)^{-1/2}\right) \times (0, t_i^1]$$

such that $|\text{Rm}_{g_i}|(x_i^2, t_i^2) > 4Q_i^1$. We compute using $Q_i^1 > 4Q_i^0$

$$\begin{aligned} d_{g_i(0)}(x_i^2, x_{0i}) &\leq d_{g_i(0)}(x_i^1, x_{0i}) + A_i(Q_i^1)^{-1/2} \\ &\leq (e^{n-1} + A_i)\epsilon_i + A_i \cdot \frac{1}{2}\epsilon_i < (2A_i + e^{n-1})\epsilon_i. \end{aligned}$$

If (x_i^2, t_i^2) satisfies the curvature bound of the claim, we choose $(\bar{x}_i, \bar{t}_i) = (x_i^2, t_i^2)$ and the claim is proved.

If (x_i^2, t_i^2) does not satisfy the claim, then there will be a point (x_i^3, t_i^3) and we can continue the above process of arguments. Hence for each i either we get a finite sequence points $\{(x_i^k, t_i^k)\}_{k=0}^{k_i}$ where $(x_i^0, t_i^0) \doteq (x_i, t_i)$ such that the claim holds by taking $(\bar{x}_i, \bar{t}_i) = (x_i^{k_i}, t_i^{k_i})$, or there is an infinite sequence of points $\{(x_i^k, t_i^k)\}_{k=0}^{\infty}$ which satisfies the following. Let $Q_i^k \doteq |\text{Rm}_{g_i}|(x_i^k, t_i^k)$, then for each integer $k \geq 0$

$$(x_i^{k+1}, t_i^{k+1}) \in B_{g_i(0)}\left(x_i^k, A_i(Q_i^k)^{-1/2}\right) \times (0, t_i^k]$$

such that $|\text{Rm}_{g_i}|(x_i^{k+1}, t_i^{k+1}) > 4Q_i^k$.

Now we show that for any i there can not be infinite sequence $\{(x_i^k, t_i^k)\}_{k=0}^\infty$ from which the claim follows. We compute

$$\begin{aligned}
& d_{g_i(0)}(x_i^{k+1}, x_{0i}) \\
& \leq d_{g_i(0)}(x_{0i}, x_i^0) + d_{g_i(0)}(x_i^0, x_i^1) + d_{g_i(0)}(x_i^1, x_i^2) + \cdots + d_{g_i(0)}(x_i^k, x_i^{k+1}) \\
& \leq e^{n-1}\epsilon_i + A_i (Q_i^0)^{-1/2} + A_i (Q_i^1)^{-1/2} + \cdots + A_i (Q_i^k)^{-1/2} \\
& \leq e^{n-1}\epsilon_i + A_i\epsilon_i + A_i\frac{1}{2}\epsilon_i + \cdots + A_i\frac{1}{2^k}\epsilon_i \\
& < (2A_i + e^{n-1})\epsilon_i,
\end{aligned}$$

where we have used $Q_i^{k+1} > 4Q_i^k > 4^{k+1}Q_i^0 > 4^{k+1}\epsilon_i^{-2}$. For any fixed i , from $A_i = \frac{1}{100n\epsilon_i}$ and $\epsilon_i \leq \frac{1}{4e^{n-1}}$, we conclude that (x_i^k, t_i^k) is in the compact set $\overline{B_{g_i(0)}(x_{0i}, 1)} \times [0, \epsilon_i^2]$ for all k . On the other hand we have

$$\lim_{k \rightarrow \infty} |\text{Rm}_{g_i}|(x_i^k, t_i^k) \geq \lim_{k \rightarrow \infty} 4^k \epsilon_i^{-2} = \infty,$$

which is impossible. Now Claim A is proved. \square

Let (\bar{x}_i, \bar{t}_i) be as given by Claim A. We divide the rest proof of Proposition 1 into three cases according to the value of

$$(3) \quad \overline{\lim}_{i \rightarrow \infty} \bar{t}_i \cdot |\text{Rm}_{g_i}|(\bar{x}_i, \bar{t}_i) \doteq \tilde{\alpha}$$

equals to infinite, positive finite number, or zero. We will derive contradictions in all three cases.

Case 1. $\tilde{\alpha} = +\infty$. From Claim A and the choice of $A_i = \frac{1}{100n\epsilon_i}$, by switching to a subsequence we have $\bar{t}_i \leq \epsilon_i^2$, $\lim_{i \rightarrow \infty} \bar{t}_i \cdot |\text{Rm}_{g_i}|(\bar{x}_i, \bar{t}_i) = \infty$, and $d_{g_i(0)}(\bar{x}_i, x_{0i}) < \frac{1}{4}$. In particular, $B_{g_i(0)}(\bar{x}_i, \frac{3}{4}) \subset B_{g_i(0)}(x_{0i}, 1)$.

From the assumption (i) and (ii) of the contradiction argument and the Bishop-Gromov volume comparison theorem there is a constant $v_1 > 0$, depending only on n and v_0 , such that

$$\text{Vol}_{g_i(0)} \left(B_{g_i(0)} \left(x_{0i}, \frac{1}{4} \right) \right) \geq v_1.$$

Since ball $B_{g_i(0)}(\bar{x}_i, \frac{1}{2})$ contains ball $B_{g_i(0)}(x_{0i}, \frac{1}{4})$ we have

$$(4) \quad \text{Vol}_{g_i(0)} \left(B_{g_i(0)} \left(\bar{x}_i, \frac{1}{2} \right) \right) \geq v_1.$$

Let $\delta \doteq \delta_0 > 0$ be the constant in Theorem 10.1 in [Pe02I] corresponding to $\alpha = 1$. Applying Lemma 1 below to metric $4g_i(0)$ and ball $B_{4g_i(0)}(\bar{x}_i, 1) = B_{g_i(0)}(\bar{x}_i, \frac{1}{2})$ we

conclude that there is a $r_1 < \frac{1}{2}$, depending only on n , δ_0 and v_1 but not depending on i , such that

$$(5) \quad \text{Area}_{g_i(0)}(\partial\Omega)^n \geq (1 - \delta_0) c_n (\text{Vol}_{g_i(0)}(\Omega))^{n-1}$$

for all regular domain $\Omega \subset B_{g_i(0)}(\bar{x}_i, r_1)$.

Let $r_2 \doteq \min\left\{r_1, \frac{1}{\sqrt{n(n-1)}}\right\}$, and let $\hat{g}_i(t) = (r_2)^{-2}g_i((r_2)^2t)$, $0 \leq t \leq (r_2)^{-2}\epsilon_i^2$. It follows from assumption (i) that the scalar curvature $R_{\hat{g}_i}(\cdot, 0) \geq -1$ on $B_{\hat{g}_i(0)}(\bar{x}_i, 1)$. It follows from (5) we have

$$\text{Area}_{\hat{g}_i(0)}(\partial\Omega)^n \geq (1 - \delta_0) c_n (\text{Vol}_{\hat{g}_i(0)}(\Omega))^{n-1}$$

for all regular domain $\Omega \subset B_{\hat{g}_i(0)}(\bar{x}_i, 1)$.

For i large enough we can apply Theorem 10.1 in [Pe02I] to $(B_{\hat{g}_i(0)}(\bar{x}_i, 1), \hat{g}_i(t))$, $0 \leq t \leq (r_2)^{-2}\epsilon_i^2$, and conclude

$$|\text{Rm}_{\hat{g}_i}|(x, t) \leq \frac{1}{t} + \frac{1}{(r_2)^{-2}\epsilon_i^2}$$

for all $t \in (0, (r_2)^{-2}\epsilon_i^2]$ and $x \in B_{\hat{g}_i(t)}(\bar{x}_i, (r_2)^{-1}\epsilon_i)$. Equivalently we have

$$|\text{Rm}_{g_i}|(x, t) \leq \frac{1}{t} + \frac{1}{\epsilon_i^2}$$

for all $t \in (0, \epsilon_i^2]$ and $x \in B_{g_i(t)}(\bar{x}_i, \epsilon_i)$. In particular

$$|\text{Rm}_{g_i}|(\bar{x}_i, \bar{t}_i) \leq \frac{1}{\bar{t}_i} + \frac{1}{\epsilon_i^2} \leq \frac{2}{\bar{t}_i}$$

for all large i . This contradicts with the assumption of Case 1 that $\tilde{\alpha}$ in (3) is infinity.

Case 2. $\tilde{\alpha} \in (0, \infty)$. Let $\hat{g}_i(t) \doteq \bar{Q}_i g_i((\bar{Q}_i)^{-1}t)$, $t \in [0, \hat{t}_i]$, where $\hat{t}_i \doteq \bar{Q}_i \bar{t}_i$. Let b_0 be a constant bigger than $\frac{11}{3}(n-1)(\tilde{\alpha}+1)+1$ to be chosen later (see (11) below). By passing to a subsequence we have

$$(2i) \quad |\text{Rm}_{\hat{g}_i}|(x, t) \leq 4 \text{ for all } x \in B_{\hat{g}_i(0)}(\bar{x}_i, A_i) \text{ and } t \in [0, \hat{t}_i].$$

$$(2ii) \quad |\text{Rm}_{\hat{g}_i}|(\bar{x}_i, \hat{t}_i) = 1.$$

$$(2iii) \quad |\text{Rm}_{\hat{g}_i}|(x, 0) \leq \bar{Q}_i^{-1} \text{ for all } x \in B_{\hat{g}_i(0)}(\bar{x}_i, A_i).$$

$$(2iv) \quad \hat{t}_i \leq \tilde{\alpha} + 1, \hat{t}_i \rightarrow \tilde{\alpha}, A_i > 2e^{4(n-1)(\tilde{\alpha}+1)}b_0, \text{ and } A_i \rightarrow \infty.$$

Applying Lemma 2 to $\hat{g}_i(t)$ with $b = b_0$ we get a function $h_i : M_i \times [0, \hat{t}_i] \rightarrow [0, 1]$ such that support

$$\text{supp } h_i(\cdot, t) \subset \bar{B}_{\hat{g}_i(t)}\left(\bar{x}_i, 2b_0 - \frac{11}{3}(n-1)t\right) \subset B_{\hat{g}_i(0)}(\bar{x}_i, A_i)$$

and

$$(6) \quad \left(\frac{\partial}{\partial t} - \Delta_{\hat{g}_i(t)} \right) h_i \leq \frac{10}{b_0^2} h_i.$$

Recall the curvature $\text{Rm}_{\hat{g}_i}$ of Ricci flow $\hat{g}_i(t)$ satisfies

$$\left(\frac{\partial}{\partial t} - \Delta_{\hat{g}_i} \right) |\text{Rm}_{\hat{g}_i}|^2 \leq -2 |\nabla_{\hat{g}_i} \text{Rm}_{\hat{g}_i}|^2 + 16 |\text{Rm}_{\hat{g}_i}|^3.$$

Now we compute the evolution equation of $h_i |\text{Rm}_{\hat{g}_i}|^2$.

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta_{\hat{g}_i} \right) \left(h_i |\text{Rm}_{\hat{g}_i}|^2 \right) \\ &= \left(\left(\frac{\partial}{\partial t} - \Delta_{\hat{g}_i} \right) h_i \right) |\text{Rm}_{\hat{g}_i}|^2 + h_i \left(\left(\frac{\partial}{\partial t} - \Delta_{\hat{g}_i} \right) |\text{Rm}_{\hat{g}_i}|^2 \right) - 2 \nabla_{\hat{g}_i} h_i \cdot \nabla_{\hat{g}_i} |\text{Rm}_{\hat{g}_i}|^2 \\ &\leq \frac{10}{b_0^2} h_i |\text{Rm}_{\hat{g}_i}|^2 + h_i \left(-2 |\nabla_{\hat{g}_i} \text{Rm}_{\hat{g}_i}|^2 + 16 |\text{Rm}_{\hat{g}_i}|^3 \right) + \frac{4\sqrt{10}}{b_0} |\text{Rm}_{\hat{g}_i}| \cdot h_i^{1/2} |\nabla_{\hat{g}_i} \text{Rm}_{\hat{g}_i}| \\ &\leq \left(\frac{10}{b_0^2} + 64 \right) h_i |\text{Rm}_{\hat{g}_i}|^2 - 2h_i |\nabla_{\hat{g}_i} \text{Rm}_{\hat{g}_i}|^2 + \frac{16\sqrt{10}}{b_0} \cdot h_i^{1/2} |\nabla_{\hat{g}_i} \text{Rm}_{\hat{g}_i}| \\ &\leq \left(\frac{10}{b_0^2} + 64 \right) \left(h_i |\text{Rm}_{\hat{g}_i}|^2 \right) + \frac{320}{b_0^2} \end{aligned}$$

where we have used

$$|\nabla_{\hat{g}_i} h_i| = \frac{|\phi'(w)|}{b_0} |\nabla_{\hat{g}_i} d_{\hat{g}_i(t)}(x, \bar{x}_i)|_{\hat{g}_i} \leq \frac{\sqrt{10}}{b_0} h_i^{1/2}$$

and $|\text{Rm}_{\hat{g}_i}| \leq 4$ on $\text{supp } h_i(\cdot, t)$. Here ϕ is the function defined in the proof of Lemma 2.

Let $u_i \doteq h_i |\text{Rm}_{\hat{g}_i}|^2$. We have proved

$$\left(\frac{\partial}{\partial t} - \Delta_{\hat{g}_i} \right) u_i \leq \left(\frac{10}{b_0^2} + 64 \right) u_i + \frac{320}{b_0^2}$$

on $M_i \times [0, \hat{t}_i]$.

Let $H_i > 0$ is the backward heat kernel to the conjugate heat equation on $(M_i, \hat{g}_i(t))$ with $t \in [0, \hat{t}_i]$ centered at \bar{x}_i , i.e.,

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \Delta_{\hat{g}_i} - R_{\hat{g}_i} \right) H_i &= 0 \\ \lim_{t \rightarrow \hat{t}_i} H_i(x, t) &= \delta_{\bar{x}_i}. \end{aligned}$$

Note that $\int_{M_i} H_i(\cdot, t) d\mu_{\hat{g}_i(t)} = 1$.

Now we compute

$$\begin{aligned}
& \frac{d}{dt} \int_{M_i} u_i H_i d\mu_{\hat{g}_i} \\
&= \int_{M_i} \left(\left(\frac{\partial}{\partial t} - \Delta_{\hat{g}_i} \right) u_i \right) H_i d\mu_{\hat{g}_i} + \int_{M_i} u_i \left(\left(\frac{\partial}{\partial t} + \Delta_{\hat{g}_i} - R_{\hat{g}_i} \right) H_i \right) d\mu_{\hat{g}_i} \\
&\leq \int_{M_i} \left(\left(\frac{10}{b_0^2} + 64 \right) u_i + \frac{320}{b_0^2} \right) H_i d\mu_{\hat{g}_i} \\
&= \left(\frac{10}{b_0^2} + 64 \right) \int_{M_i} u_i H_i d\mu_{\hat{g}_i} + \frac{320}{b_0^2}.
\end{aligned}$$

Hence it follows from a simple integration that $U_i(t) \doteq \int_{M_i} u_i H_i d\mu_{\hat{g}_i}$ satisfies

$$(7) \quad U_i(t) \leq e^{\left(\frac{10}{b_0^2} + 64\right)t} U_i(0) + \frac{320}{\left(\frac{10}{b_0^2} + 64\right) b_0^2} \left(e^{\left(\frac{10}{b_0^2} + 64\right)t} - 1 \right)$$

for $t \in [0, \hat{t}_i]$.

By the definition of h_i we have at $t = \hat{t}_i$

$$(8) \quad U_i(\hat{t}_i) = u_i(\bar{x}_i, \hat{t}_i) = \phi\left(\frac{\frac{11}{3}(n-1)\hat{t}_i}{b_0}\right) |\text{Rm}_{\hat{g}_i}|^2(\bar{x}_i, \hat{t}_i) = 1.$$

On the other hand we have

$$\begin{aligned}
U_i(0) &= \int_{M_i} h_i(x, 0) |\text{Rm}_{\hat{g}_i}|^2(x, 0) H_i(x, 0) d\mu_{\hat{g}_i(0)} \\
&\leq \int_{B_{\hat{g}_i(0)}(\bar{x}_i, 2b_0)} |\text{Rm}_{\hat{g}_i}|^2(x, 0) H_i(x, 0) d\mu_{\hat{g}_i(0)} \\
&\leq \bar{Q}_i^{-2} \int_{B_{\hat{g}_i(0)}(\hat{x}_i, 2b_0)} H_i(x, 0) d\mu_{\hat{g}_i(0)} \\
&\leq \bar{Q}_i^{-2} \int_{M_i} H_i(x, 0) d\mu_{\hat{g}_i(0)}
\end{aligned}$$

where we have used $\text{supp } h_i(\cdot, 0) \subset B_{\hat{g}_i(0)}(\bar{x}_i, 2b_0)$ in the first inequality and (2iii) in the second inequality. Hence we have

$$(9) \quad U_i(0) \leq \bar{Q}_i^{-2}.$$

By combining (7), (8), and (9) we get

$$1 \leq e^{\left(\frac{10}{b_0^2} + 64\right)\hat{t}_i} \bar{Q}_i^{-2} + \frac{320}{\left(\frac{10}{b_0^2} + 64\right) b_0^2} \left(e^{\left(\frac{10}{b_0^2} + 64\right)\hat{t}_i} - 1 \right).$$

Hence

$$(10) \quad 1 \leq e^{\left(\frac{10}{b_0^2}+64\right)(\tilde{\alpha}+1)} \bar{Q}_i^{-2} + \frac{320}{\left(\frac{10}{b_0^2}+64\right)b_0^2} e^{\left(\frac{10}{b_0^2}+64\right)(\tilde{\alpha}+1)}.$$

Let

$$(11) \quad b_0 \doteq \max\left\{\frac{11}{3}(n-1)(\tilde{\alpha}+1)+1, 3e^{33(\tilde{\alpha}+1)}\right\}$$

For such choice of b_0 we have $\frac{320}{\left(\frac{10}{b_0^2}+64\right)b_0^2} e^{\left(\frac{10}{b_0^2}+64\right)(\tilde{\alpha}+1)} < \frac{5}{9}$. (10) is impossible since

$\bar{Q}_i \rightarrow \infty$. Hence we get the required contradiction for Case 2.

Case 3. $\tilde{\alpha} = 0$. The proof for this case is similar to the proof of Case 2. Let $\hat{g}_i(t) \doteq \bar{Q}_i g_i((\bar{Q}_i)^{-1}t)$, $t \in [0, \hat{t}_i]$, where $\hat{t}_i \doteq \bar{Q}_i \bar{t}_i$. By passing to a subsequence we have

$$(3i) \quad |\text{Rm}_{\hat{g}_i}|(x, t) \leq 4 \text{ for all } x \in B_{\hat{g}_i(0)}(\bar{x}_i, A_i) \text{ and } t \in [0, \hat{t}_i].$$

$$(3ii) \quad |\text{Rm}_{\hat{g}_i}|(\bar{x}_i, \hat{t}_i) = 1.$$

$$(3iii) \quad |\text{Rm}_{\hat{g}_i}|(x, 0) \leq \bar{Q}_i^{-1} \text{ for all } x \in B_{\hat{g}_i(0)}(\bar{x}_i, A_i).$$

$$(3iv) \quad \hat{t}_i \leq \frac{1}{6(n-1)}, \hat{t}_i \rightarrow 0, A_i > 4e^2, \text{ and } A_i \rightarrow \infty.$$

Applying Lemma 2 to $\hat{g}_i(t)$ with $b = 2$ we get a function $h_i : M_i \times [0, \hat{t}_i] \rightarrow [0, 1]$ such that support

$$\text{supp } h_i(\cdot, t) \subset \bar{B}_{\hat{g}_i(t)}\left(\bar{x}_i, 4 - \frac{11}{3}(n-1)t\right) \subset B_{\hat{g}_i(0)}(\bar{x}_i, A_i)$$

and

$$(12) \quad \left(\frac{\partial}{\partial t} - \Delta_{\hat{g}_i(t)}\right) h_i \leq \frac{5}{2} h_i.$$

We compute

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta_{\hat{g}_i}\right) \left(h_i |\text{Rm}_{\hat{g}_i}|^2\right) \\ &= \left(\left(\frac{\partial}{\partial t} - \Delta_{\hat{g}_i}\right) h_i\right) |\text{Rm}_{\hat{g}_i}|^2 + h_i \left(\left(\frac{\partial}{\partial t} - \Delta_{\hat{g}_i}\right) |\text{Rm}_{\hat{g}_i}|^2\right) - 2\nabla_{\hat{g}_i} h_i \cdot \nabla_{\hat{g}_i} |\text{Rm}_{\hat{g}_i}|^2 \\ &\leq \frac{5}{2} h_i |\text{Rm}_{\hat{g}_i}|^2 + h_i \left(-2|\nabla_{\hat{g}_i} \text{Rm}_{\hat{g}_i}|^2 + 16|\text{Rm}_{\hat{g}_i}|^3\right) + 2\sqrt{10} |\text{Rm}_{\hat{g}_i}| \cdot h_i^{1/2} |\nabla_{\hat{g}_i} \text{Rm}_{\hat{g}_i}| \\ &\leq \frac{133}{2} h_i |\text{Rm}_{\hat{g}_i}|^2 - 2h_i |\nabla_{\hat{g}_i} \text{Rm}_{\hat{g}_i}|^2 + 8\sqrt{10} h_i^{1/2} |\nabla_{\hat{g}_i} \text{Rm}_{\hat{g}_i}| \\ &\leq \frac{133}{2} \left(h_i |\text{Rm}_{\hat{g}_i}|^2\right) + 80 \end{aligned}$$

where we have used

$$|\nabla_{\hat{g}_i} h_i| = \frac{|\phi'(w)|}{2} |\nabla_{\hat{g}_i} d_{\hat{g}_i(t)}(x, \bar{x}_i)|_{\hat{g}_i} \leq \frac{\sqrt{10}}{2} h_i^{1/2}$$

and $|\text{Rm}_{\hat{g}_i}| \leq 4$ on $\text{supp } h_i(\cdot, t)$.

Let $u_i \doteq h_i |\text{Rm}_{\hat{g}_i}|^2$. We have proved

$$\left(\frac{\partial}{\partial t} - \Delta_{\hat{g}_i} \right) u_i \leq 67u_i + 80$$

on $M_i \times [0, \hat{t}_i]$.

Let $H_i > 0$ is the backward heat kernel to the conjugate heat equation on $(M_i, \hat{g}_i(t))$ with $t \in [0, \hat{t}_i]$ centered at \bar{x}_i which satisfies $\int_{M_i} H_i(\cdot, t) d\mu_{\hat{g}_i(t)} = 1$. We compute

$$\begin{aligned} \frac{d}{dt} \int_{M_i} u_i H_i d\mu_{\hat{g}_i} &= \int_{M_i} \left(\left(\frac{\partial}{\partial t} - \Delta_{\hat{g}_i} \right) u_i \right) H_i d\mu_{\hat{g}_i} \\ &\leq \int_{M_i} (67u_i + 80) H_i d\mu_{\hat{g}_i} \\ &= 67 \int_{M_i} u_i H_i d\mu_{\hat{g}_i} + 80. \end{aligned}$$

Hence it follows from a simple integration that $U_i(t) \doteq \int_{M_i} u_i H_i d\mu_{\hat{g}_i}$ satisfies

$$(13) \quad U_i(t) \leq e^{67t} U_i(0) + \frac{80}{67} (e^{67t} - 1)$$

for $t \in [0, \hat{t}_i]$.

At $t = \hat{t}_i$ we have

$$(14) \quad U_i(\hat{t}_i) = u_i(\bar{x}_i, \hat{t}_i) = \phi\left(\frac{\frac{11}{3}(n-1)\hat{t}_i}{2}\right) |\text{Rm}_{\hat{g}_i}|^2(\bar{x}_i, \hat{t}_i) = 1.$$

On the other hand by an argument similar to the proof of (9) we have

$$(15) \quad U_i(0) \leq \bar{Q}_i^{-2}.$$

By combining (13), (14), and (15) we get

$$1 \leq e^{67\hat{t}_i} \bar{Q}_i^{-2} + \frac{80}{67} (e^{67\hat{t}_i} - 1).$$

This is impossible since $\hat{t}_i \rightarrow 0$ and $\bar{Q}_i \rightarrow \infty$. Hence we get the required contradiction for Case 3.

Now we have finished the proof of Proposition 1. \square

3 Proof of two technical lemmas

In the proof of Proposition 1 we have used the following two lemmas. Intuitively the first lemma says that if a ball of radius 1 has bounded sectional curvature and is noncollapsing, then the isoperimetric constant on small certain size ball is close to the Euclidean one. Note that the next lemma and essential the same proof are also given by Yuanqi Wang [W].

Lemma 1 *Given $n, v_0 > 0$, and $\delta_0 > 0$, there is $r > 0$, depending only on n, v_0 , and δ_0 , which has the following property. Let $B(x_0, 1)$ be a ball in Riemannian manifold (M^n, g) which satisfies*

- (I) *The closure $\bar{B}(x_0, 1)$ is compact in M ;*
- (II) *The Riemann curvature $|\text{Rm}| \leq 1$ on $B(x_0, 1)$, and*
- (III) *The volume $\text{Vol}(B(x_0, 1)) \geq v_0 > 0$.*

Then we have

$$(16) \quad \text{Area}(\partial\Omega)^n \geq (1 - \delta_0) c_n (\text{Vol}(\Omega))^{n-1}$$

for all regular domain $\Omega \subset B(x_0, r)$. Here $c_n = n^n \omega_n$ is the isoperimetric constant for Euclidean space.

Proof. Step 1. Injectivity radius bound. Under the assumption of Lemma 1, by a theorem of Cheeger-Gromov-Taylor ([CGT], Theorem A.7) there is a $\iota_0 > 0$ depending only on n and v_0 such that for any $x \in B(x_0, \frac{1}{2})$ we have $\text{inj}_x \geq \iota_0$.

Step 2. Metric tensor on ball $B(x_0, 1)$. Let $x = (x^i)$ be the normal coordinates at x_0 . It follows from a result of Hamilton ([CCCY], Theorem 4.10 on p.308) that for any $\varepsilon > 0$ there is $\lambda_0 = \lambda_0(n, \varepsilon)$ such that metric tensor

$$(17) \quad (1 - \varepsilon) \delta_{ij} \leq g_{ij} \leq (1 + \varepsilon) \delta_{ij}.$$

for all $|x| \leq \lambda_0$. Note that (δ_{ij}) is Euclidean metric in the coordinates (x^i) .

Step 3. Approximation argument. Let $r \doteq \min\{\iota_0, \lambda_0\}$. Now we consider a regular domain $\Omega \subset B(x_0, r)$. We compute

$$\begin{aligned} \text{Vol}_g(\Omega) &= \int_{\Omega} \sqrt{\det(g_{ij})} \cdot dx^1 \dots dx^n \\ &\leq \int_{\Omega} \sqrt{(1 + \varepsilon)^n \det(\delta_{ij})} \cdot dx^1 \dots dx^n \\ &= (1 + \varepsilon)^{n/2} \text{Vol}_{\text{Euc}}(\Omega). \end{aligned}$$

Let $\{\theta_a\}_{a=1}^{n-1}$ be an orthonormal frame of $(\partial\Omega, (\delta_{ij})|_{\partial\Omega})$ at some point x and let $\{\theta_a^*\}$ be the dual frame. The area form $d\sigma_{(\partial\Omega, (\delta_{ij})|_{\partial\Omega})}$ at x is given by $\theta_1^* \wedge \dots \wedge \theta_{n-1}^*$. The

area form $d\sigma_{(\partial\Omega, g|_{\partial\Omega})}$ at x is given by $\sqrt{\det(g(\theta_a, \theta_b))_{(n-1) \times (n-1)}} \cdot \theta_1^* \wedge \dots \wedge \theta_{n-1}^*$. We can estimate

$$\sqrt{\det(g(\theta_a, \theta_b))_{(n-1) \times (n-1)}} \geq \sqrt{(1-\epsilon)^{n-1} \det((\delta_{ij})(\theta_a, \theta_b))} = (1-\epsilon)^{(n-1)/2},$$

hence

$$\begin{aligned} \text{Area}_{g|_{\partial\Omega}}(\partial\Omega) &= \int_{\partial\Omega} d\sigma_{(\partial\Omega, g|_{\partial\Omega})} \\ &\geq \int_{\partial\Omega} (1-\epsilon)^{(n-1)/2} d\sigma_{(\partial\Omega, (\delta_{ij})|_{\partial\Omega})} \\ &= (1-\epsilon)^{(n-1)/2} \text{Area}_{\text{Euc}}(\partial\Omega). \end{aligned}$$

Now we compute

$$\begin{aligned} \frac{(\text{Area}_{g|_{\partial\Omega}}(\partial\Omega))^n}{(\text{Vol}_g(\Omega))^{n-1}} &\geq \frac{((1-\epsilon)^{(n-1)/2} \text{Area}_{\text{Euc}}(\partial\Omega))^n}{((1+\epsilon)^{n/2} \text{Vol}_{\text{Euc}}(\Omega))^{n-1}} \\ &= \left(\frac{1-\epsilon}{1+\epsilon}\right)^{\frac{(n-1)n}{2}} \cdot \frac{(\text{Area}_{\text{Euc}}(\partial\Omega))^n}{(\text{Vol}_{\text{Euc}}(\Omega))^n} \\ &\geq \left(\frac{1-\epsilon}{1+\epsilon}\right)^{\frac{(n-1)n}{2}} c_n. \end{aligned}$$

Given δ_0 we choose ϵ such that

$$\left(\frac{1-\epsilon}{1+\epsilon}\right)^{\frac{(n-1)n}{2}} = 1 - \delta_0,$$

this in turn requires us to choose the corresponding $\lambda_0(n, \epsilon)$ to ensure (17). Then Lemma 1 holds for $r = \min\{\iota_0, \lambda_0\}$. \square

The second lemma is about the existence of an auxiliary function.

Lemma 2 *Let $(M^n, g(t))$, $t \in [0, \hat{t}]$, be a solution of Ricci flow. Let b be a constant bigger than $\frac{11}{3}(n-1)\hat{t} + 1$ and let A be a constant bigger or equal to $2e^{4(n-1)\hat{t}}b$. We assume that closed ball $\overline{B}_{g(0)}(\bar{x}, A) \subset M$ be a compact subset and that $|\text{Rm}|(x, t) \leq 4$ for all $(x, t) \in B_{g(0)}(\bar{x}, A) \times [0, \hat{t}]$. Then there is a function $h : \overline{B}_{g(0)}(\bar{x}, A) \times [0, \hat{t}] \rightarrow [0, 1]$ such that for each $t \in [0, \hat{t}]$ the support*

$$\text{supp } h(\cdot, t) \subset \overline{B}_{g(t)}(\bar{x}, 2b - \frac{11}{3}(n-1)t) \subset B_{g(0)}(\bar{x}, A)$$

and for all $(x, t) \in M \times [0, \hat{t}]$

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) h \leq \frac{10}{b^2} h.$$

Proof. Let $\phi : \mathbb{R} \rightarrow [0, 1]$ be a smooth function which is strictly decreasing on the interval $[1, 2]$ and which satisfies

$$(18) \quad \phi(s) = \begin{cases} 1 & \text{if } s \in (-\infty, 1], \\ 0 & \text{if } s \in [2, \infty), \end{cases}$$

and

$$(19a) \quad (\phi'(s))^2 \leq 10\phi(s),$$

$$(19b) \quad \phi''(s) \geq -10\phi(s)$$

for $s \in \mathbb{R}$. We define for any $t \in [0, T]$

$$h(x, t) = \phi\left(\frac{d_{g(t)}(x, \bar{x}) + at}{b}\right)$$

where a and b are two positive constants to be chosen. Note that $\text{supp } h(\cdot, t) \subset B_{g(t)}(\bar{x}, 2b - at)$.

By the curvature assumption we have $B_{g(t)}(\bar{x}, e^{-4(n-1)t}A) \subset B_{g(0)}(\bar{x}, A)$ for all $t \in [0, \hat{t}]$. We choose $2b \leq e^{-4(n-1)\hat{t}}A$ so that $\text{supp } h(\cdot, t) \subset B_{g(0)}(\bar{x}, A)$.

Let $w(x, t) \doteq \frac{d_{g(t)}(x, \bar{x}) + at}{b}$. We compute

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) h \\ &= \frac{\phi'(w)}{b} \left(\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) d_{g(t)}(x, \bar{x}) + a \right) - \frac{\phi''(w)}{b^2} |\nabla_{g(t)} d_{g(t)}(x, \bar{x})|_{g(t)}^2 \\ &\leq \frac{\phi'(w)}{b} \left(\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) d_{g(t)}(x, \bar{x}) + a \right) + \frac{10}{b^2} h. \end{aligned}$$

Choosing a such that $a\hat{t} < b - 1$, then for $x \in B_{g(t)}(\bar{x}, 1)$ or $x \notin \text{supp } h(\cdot, t)$ we have $\phi'(w)(x, t) = 0$. Hence for such x we have

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) h \leq \frac{10}{b^2} h.$$

For $x \notin B_{g(t)}(\bar{x}, 1)$ and $x \in \text{supp } h(\cdot, t)$, we use Lemma 8.3(a) in [Pe02I] with $r_0 = 1$ and $K = 4$ and get

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) d_{g(t)}(x, \bar{x}) \Big|_{t=t_0} \geq -(n-1) \left(\frac{2}{3} K r_0 + \frac{1}{r_0} \right) = -\frac{11}{3}(n-1).$$

By choosing $a \doteq \frac{11}{3}(n-1)$ we obtain using $\phi'(w) \leq 0$

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) h \leq \frac{10}{b^2} h.$$

The lemma is proved. \square

4 An example. In this section we give an example showing that the volume lower bound assumption in Theorem 2 can not be dropped. Let r be an arbitrary positive constant. Let (Σ^2, g_r^0) be a sphere which contains a round cylinder $S^1(r) \times [-1, 1]$ of radius r and length 2. We have $\text{Vol}_{g_r^0}(\Sigma) \geq 4\pi r$. We assume volume $\text{Vol}_{g_r^0}(\Sigma) \leq 20r$. Let $(\Sigma^2, g_r(t))$, $t \in [0, T_r)$, be the maximal solution of Ricci flow with $g_r(0) = g_r^0$. Then

$$T_r = \frac{1}{8\pi} \text{Vol}_{g_r^0}(\Sigma) \in \left(\frac{1}{2}r, \frac{5}{2\pi}r \right].$$

Let $p \in S^1(r)$. Then $x_0 \doteq (p, 0)$ is a point in Σ . For any ϵ_0 we can choose r small enough so that $T_r < \epsilon_0$. Clearly we have $|\text{Rm}|(x, 0) = 0$ for all $x \in B_{g_r(0)}(x_0, 1)$ and $\text{Vol}_{g_r^0}(B_{g_r(0)}(x_0, 1)) \leq 4\pi r$. For any $\epsilon \in (\frac{1}{2}r, T_r)$, should the conclusion of Theorem 2 holds for $g_r(t)$, we would have $|\text{Rm}|(x_0, \epsilon) \leq \epsilon^{-2} < 4r^{-2}$. Since ϵ is arbitrary, we have $\lim_{t \rightarrow T_r} |\text{Rm}|(x_0, t) < 4r^{-2}$. However it is well know that limit should be infinity. Hence Theorem 2 does not hold for $g_r(t)$.

By taking the product of $(\Sigma^2, g_r(t))$ with flat torus we get high dimensional examples.

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References

- [CCCY] H.D. CAO, B. CHOW, S.C. CHU, AND S.T. YAU, editors. *Collected papers on Ricci flow*. Internat. Press, Somerville, MA, 2003.
- [CGT] J. CHEEGER, M. GROMOV, AND M. TAYLOR, *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds*. J. Diff. Geom. **17** (1982), 15-53.
- [Pe02I] G. PERELMAN, *The entropy formula for the Ricci flow and its geometric applications*. ArXiv: math.DG/0211159.
- [W] Y. WANG, *Pseudolocality of Ricci Flow under Integral Bound of Curvature*. arXiv:0903.2913

PENG LU, University of Oregon
e-mail: penglu@uoegon.edu